Putnam training problems

2017 - Set 8

Problem 1 *A Hadamard Matrix is a an $n \times n$ matrix such that every entry is 1 or -1 and every pair of columns is orthogonal. Prove that if A is a Hadamard matrix and $n > 2$ then n is a multiple of 4.*

Problem 2 *Do there exist $n \times n$ matrices A, B such that $AB - BA = I_n$?*

Problem 3 *Let $\alpha_1, \dots, \alpha_n$ be real numbers. Find the determinant of the $n \times n$ matrix*

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \dots & \alpha_n^{n-1} \end{pmatrix}$$

Problem 4 *Find the determinant of the matrix*

$$\begin{pmatrix} (1+x^2)^2 & (1+xy)^2 & (1+xz)^2 \\ (1+xy)^2 & (1+y^2)^2 & (1+yz)^2 \\ (1+xz)^2 & (1+yz)^2 & (1+z^2)^2 \end{pmatrix}$$

Problem 5 *Compute the determinan of the $n \times n$ matrix $A = (a_{ij})_{ij}$ where*

$$a_{ij} = \begin{cases} (-1)^{|i-j|} & \text{if } i \neq j \\ 2 & \text{if } i = j. \end{cases}$$

Problem 6 *Let $A = (a_{ij})$ be an $n \times n$ matrix such that*

$$\sum_{j=1}^{n} |a_{ij}| < 1$$

for all i. Prove that $I_n - A$ is invertible.

Problem 7 *Prove that in $\mathbb{R}^n$ there is no set of $d+2$ vectors whose pairwise angles are obtuse.*

Problem 8 *Let X be a set of n elements and $X_1, \dots, X_{n+1}$ be non-empty subsets of X. Prove that we can find find two non-empty families $\mathcal{A}, \mathcal{B}$ of the X_i who don't share any X_i but such that $\cup\mathcal{A} = \cup\mathcal{B}$.*

Problem 9 *There are n persons in a town. They decide to form clubs (persons may be in more than one club). They notice that every club has an odd number of persons and any two clubs share an even number of persons. What is the largest number of clubs there can be?*